SUBJECT: MATHEMATICS

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① Solve the homogeneous Fredholm integral equation

$$\phi(x) = \lambda \int_0^1 e^x e^\xi \phi(\xi) d\xi$$

<u>Sol:</u> Given

$$\phi(x) = \lambda \int_0^1 e^x e^\xi \phi(\xi) d\xi \quad \text{———} \quad (1)$$

$$\phi(x) = \lambda e^x . c_1 \quad \text{———} \quad (2)$$

$$c_1 = \int_0^1 e^\xi \phi(\xi) d\xi \quad \text{———} \quad (3)$$

From (2), $\phi(\xi) = \lambda c_1 e^\xi$ ——— (4)

Equ's (3) and (4) ⇒

$$c_1 = \int_0^1 e^\xi [\lambda c_1 e^\xi] d\xi = \lambda c_1 \left[\frac{e^{2\xi}}{2} \right]_0^1$$

$$c_1 = \frac{\lambda c_1}{2} [e^2 - 1] \Rightarrow \lambda = \frac{2}{e^2-1} \quad \text{if } c_1 \neq 0$$

∴ $\phi(x) = \lambda e^x c_1$

$$\phi(x) = \frac{2}{e^2-1} . e^x . c_1$$

$$\Rightarrow \phi(x) = e^x \quad \text{if} \quad \frac{2c_1}{e^2-1} = 1 \quad \text{i.e. } c_1 = \frac{e^2-1}{2}$$

∴ $\phi(x) = e^x$ is an eigen fn corresponding to eigen value $\lambda = \frac{2}{e^2-1}$.

② solve $\phi(x) = e^x + \lambda \int_0^1 (5x^2-3)\xi^2 \phi(\xi) d\xi$

solution. Given

$$\phi(x) = e^x + \lambda \int_0^1 (5x^2-3)\xi^2 \phi(\xi) d\xi \quad \text{—} \quad (1)$$

Here $f(x) = e^x$

$$\phi(x) = e^x + \lambda(5x^2-3) c_1 \quad \text{———} \quad (2)$$

where

$$c_1 = \int_0^1 \xi^2 \phi(\xi) d\xi \quad \text{———} \quad (3)$$

From (2)

$$\phi(\xi) = e^{\xi} + \lambda c_1 (5\xi^2 - 3) \quad \text{——— (4)}$$

From (3) and (4)

$$c_1 = \int_0^1 \xi^2 [e^{\xi} + \lambda c_1 (5\xi^2 - 3)] d\xi$$

$$c_1 = \int_0^1 \xi^2 e^{\xi} d\xi + 5\lambda c_1 \int_0^1 \xi^4 d\xi - 3\lambda c_1 \int_0^1 \xi^2 d\xi$$

$$c_1 = \xi^2 e^{\xi}\Big|_0^1 - \int_0^1 2\xi e^{\xi} d\xi + \cancel{\lambda c_1} - \cancel{\lambda c_1}$$

$$c_1 = e - 2\left[(\xi e^{\xi})\Big|_0^1 - \int_0^1 e^{\xi} d\xi\right]$$

$$c_1 = e - 2[e - (e-1)]$$

$$c_1 = (e-2)$$

in Eqn (2) $\Rightarrow$

$$\phi(x) = e^{x} + \lambda c_1 (5x^2 - 3), \quad c_1 = (e-2)$$

is the required solution.

(3) Solve

$$\phi(x) = 2x + \lambda \int_0^1 \sin(\log x)\, \phi(\xi)\, d\xi$$

Solution, Given

$$\phi(x) = 2x + \lambda \int_0^1 \sin(\log x)\, \phi(\xi)\, d\xi \quad \text{——— (1)}$$

$$\Rightarrow \phi(x) = 2x + \lambda \sin(\log x) \cdot c_1 \quad \text{——— (2)}$$

where

$$c_1 = \int_0^1 \phi(\xi)\, d\xi \quad \text{——— (3)}$$

from (2), we have

$$\phi(\xi) = 2\xi + \lambda c_1 \sin(\log \xi) \quad \text{——— (4)}$$

Eqn (3) & (4) $\Rightarrow$

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$$c_1 = \int_0^1 [2\xi + \lambda c_1 \sin(\log \xi)]\, d\xi$$

$$c_1 = \int_0^1 2\xi \, d\xi + \lambda c_1 \int_0^1 \sin(\log \xi)\, d\xi$$

$\log \xi = t$
$\Rightarrow \xi = e^t$
$\Rightarrow d\xi = e^t dt$

$$c_1 = 1 + \lambda c_1 \int \sin t \, . \, e^t dt$$

$$c_1 = 1 + \lambda c_1 \left[\frac{e^t}{2} [\sin t - \cos t] \right]$$

$$c_1 = 1 + \frac{\lambda c_1}{2} \left[e^{\log \xi} [\sin(\log \xi) - \cos(\log \xi)] \right]_0^1$$

$$c_1 = 1 + \frac{\lambda c_1}{2} [(0 - 1)]$$

$$c_1 \left(1 + \frac{\lambda}{2}\right) = 1 \Rightarrow c_1 = \frac{2}{2+\lambda}$$

$\therefore$ Required solution is given by

$$\phi(x) = 2x + \lambda c_1 \sin(\log x)$$

$$\Rightarrow \phi(x) = 2x + \lambda . \frac{2}{2+\lambda} \sin(\log x), \text{ if } \lambda \neq -2.$$

If $\lambda = -2$,

$$F = \int_0^1 f(x) \bar{\phi}(x)\, dx,$$

clearly $F \neq 0 \Rightarrow \exists$ no solution.

If $\lambda = -2$, $F = 0 \Rightarrow \exists$ infinitely many solutions & given by

$$\phi(x) = 2x + c \sin(\log x),$$ c is arbitrary.

Solution of non homogeneous Volterra's Integral equation by Laplace transform technique.

(1) Solve

$$\phi(x) = 1 - \int_0^x (x-\xi)\,\phi(\xi)\,d\xi \qquad (\text{Form } (x-\xi))$$

$$\Rightarrow \phi(x) = 1 - \phi(x) * x \qquad \left[\text{Using convolution theorem } \int_0^t (t-\xi)\,\phi(\xi)\,d\xi = \phi(t) * t\right]$$

Taking Laplace transform, we have

$$\bar{\phi}(p) = \frac{1}{p} - L[\phi(x)] \cdot L(x)$$

$$\bar{\phi}(p) = \frac{1}{p} - \bar{\phi}(p) \cdot \frac{1}{p^2}$$

$$\Rightarrow \left(1 + \frac{1}{p^2}\right)\bar{\phi}(p) = \frac{1}{p} \Rightarrow \frac{p^2+1}{p^2}\,\bar{\phi}(p) = \frac{1}{p}$$

$$\Rightarrow \bar{\phi}(p) = \frac{p}{p^2+1}$$

Taking inverse Laplace transform, we have

$$\phi(x) = L^{-1}\left(\frac{p}{p^2+1}\right) = \cos x$$

$\Rightarrow \phi(x) = \cos x$ is the solution of the given integral equation.

(2) Solve and verify your result.

$$\phi(x) = 1 + \int_0^x \sin(x-\xi)\,\phi(\xi)\,d\xi$$

Solution. Given $\phi(x) = 1 + \int_0^x \sin(x-\xi)\,\phi(\xi)\,d\xi$ —(1)

Taking Laplace transform

$$\bar{\phi}(p) = \frac{1}{p} + L[\sin x * \phi(x)] \quad [\text{by convolution theorem}]$$

$$= \frac{1}{p} + L(\sin x) \cdot L(\phi(x)) \qquad \left[\because \int_0^x \sin(x-\xi)\,\phi(\xi)\,d\xi = \sin x * \phi(x)\right]$$

$$\bar{\phi}(p) = \frac{1}{p} + \frac{1}{p^2+1} \cdot \bar{\phi}(p)$$

$$\left(1-\frac{1}{p^2+1}\right)\bar{\phi}(p) = \frac{1}{p}$$

$$\Rightarrow \quad \frac{p^2}{p^2+1}\bar{\phi}(p) = \frac{1}{p}$$

$$\Rightarrow \quad \bar{\phi}(p) = \frac{p^2+1}{p^3} = \frac{1}{p}+\frac{1}{p^3}$$

Taking inverse Laplace transform, we have

$$\phi(x) = L^{-1}\left(\frac{1}{p}\right) + L^{-1}\left(\frac{1}{p^3}\right) \qquad L^{-1}\left(\frac{1}{p^n}\right) = \frac{x^{n-1}}{\lfloor n-1}$$

$$\phi(x) = 1+\frac{x^2}{2}$$

<u>Verification</u>. To show $\phi(x) = \left(1+\frac{x^2}{2}\right)$ is a solution of $\phi(x) = 1+\int_0^x \sin(x-\xi)\phi(\xi)d\xi$

$$RHS = 1+\int_0^x \sin(x-\xi)\,\phi(\xi)d\xi \qquad \left[\begin{array}{l} \because \phi(x) = 1+\frac{x^2}{2} \\ \Rightarrow \phi(\xi) = 1+\frac{\xi^2}{2} \end{array}\right]$$

$$RHS = 1+\int_0^x \underbrace{\sin(x-\xi)}_{II}\underbrace{\left[1+\frac{\xi^2}{2}\right]}_{I} d\xi$$

$$RHS = 1+\left(1+\frac{\xi^2}{2}\right).\cos(x-\xi)\Big|_0^x - \int_0^x \xi.\cos(x-\xi)\,d\xi$$

$$RHS = 1+\left(1+\frac{x^2}{2}\right) - \cos x - \left[-\xi.\sin(x-\xi)\Big|_0^x + \int_0^x \sin(x-\xi)\,d\xi\right.$$

$$RHS = 1+1+\frac{x^2}{2} - \cos x + \left[-\cos(x-\xi)\right]_0^x$$

$$RHS = 2+\frac{x^2}{2} - \cos x - \left[1-\cos x\right]$$

$$RHS = 2+\frac{x^2}{2} - \cancel{\cos x} - 1 + \cancel{\cos x}$$

$$RHS = 1+\frac{x^2}{2}.$$

RHS = LHS, So result is verified.

(3) Solve and Verify

$$\phi(x) = x^2 + \int_0^x \sin(x-\xi)\,\phi(\xi)\,d\xi$$

Solⁿ. Given

$$\phi(x) = x^2 + \int_0^x \sin(x-\xi)\,\phi(\xi)\,d\xi \quad \text{———} (1)$$

$\phi(x) = x^2 + \sin x * \phi(x)$, by convolution theorem

Taking Laplace transform of both sides of (i), we have

$$\bar{\phi}(p) = L(x^2) + L(\sin x * \phi(x))$$

$$\bar{\phi}(p) = \frac{\lfloor 2}{p^3} + L(\sin x) \cdot L[\phi(x)]$$

$$\bar{\phi}(p) = \frac{2}{p^3} + \frac{1}{p^2+1} \cdot \bar{\phi}(p)$$

$$\Rightarrow \left(1 - \frac{1}{p^2+1}\right)\bar{\phi}(p) = \frac{2}{p^3}.$$

$$\Rightarrow \frac{p^2}{p^2+1}\bar{\phi}(p) = \frac{2}{p^3}$$

$$\bar{\phi}(p) = \frac{2(p^2+1)}{p^5} = \frac{2}{p^3} + \frac{2}{p^5}.$$

Taking inverse Laplace transform, we have

$$\phi(x) = 2L^{-1}\left(\frac{1}{p^3}\right) + 2L^{-1}\left[\frac{1}{p^5}\right]$$

$$= 2\left(\frac{x^2}{\lfloor 2}\right) + 2\left(\frac{x^4}{\lfloor 4}\right)$$

$\therefore \phi(x) = x^2 + \dfrac{x^4}{12}$ which is a solⁿ of (1)

Verification. To show $\phi(x) = x^2 + \dfrac{x^4}{12}$ is a solution of

$$\phi(x) = x^2 + \int_0^x \sin(x-\xi)\,\phi(\xi)\,d\xi$$

$$RHS = x^2 + \int_0^x \sin(x-\xi)\,\phi(\xi)\,d\xi$$

$$RHS = x^2 + \int_0^x \sin(x-\xi)\left\{\xi^2 + \frac{\xi^4}{12}\right\}d\xi$$

$$\left[\because \phi(x) = x^2 + \frac{x^4}{12} \Rightarrow \phi(\xi) = \xi^2 + \frac{\xi^4}{12}\right]$$

$$RHS = x^2 + \int_0^x \xi^2 \sin(x-\xi)\,d\xi + \int_0^x \frac{\xi^4}{12}\sin(x-\xi)\,d\xi$$

$$RHS = x^2 + \int_0^x \xi^2 \sin(x-\xi)\,d\xi + \frac{1}{12}\left[\xi^4\cos(x-\xi)\Big|_0^x - 4\int_0^x \xi^3\cos(x-\xi)\,d\xi\right]$$

$$RHS = x^2 + \int_0^x \xi^2 \sin(x-\xi)\,d\xi + \frac{1}{12}\left[x^4\right] - \frac{4}{12}\int_0^x \xi^3\cos(x-\xi)\,d\xi$$

$$RHS = x^2 + \frac{x^4}{12} + \int_0^x \xi^2 \sin(x-\xi)\,d\xi - \frac{1}{3}\left[-\xi^3\sin(x-\xi)\Big|_0^x + \int_0^x 3\xi^2\sin(x-\xi)\,d\xi\right]$$

$$RHS = x^2 + \frac{x^4}{12} + \int_0^x \xi^2 \underline{\sin(x-\xi)\,d\xi} + 0 - \underline{\int_0^x \xi^2\sin(x-\xi)\,d\xi}$$

$$RHS = x^2 + \frac{x^4}{12}$$

$$RHS = LHS.$$

(4) Solve $\phi(x) = x + 2\int_0^x \cos(x-\xi)\phi(\xi)\,d\xi$

Solution. Given

$$\phi(x) = x + 2\int_0^x \cos(x-\xi)\phi(\xi)\,d\xi \quad \text{— (1)}$$

$\phi(x) = x + 2\cos x * \phi(x)$, by convolution Theorem

Taking Laplace transform of both sides of (1), We get

$$\bar{\phi}(p) = \frac{1}{p^2} + 2\cdot\frac{p}{p^2+1}\bar{\phi}(p)$$

$$\Rightarrow \left(1 - \frac{2p}{p^2+1}\right)\bar{\phi}(p) = \frac{1}{p^2}$$

$$\Rightarrow \frac{p^2+1-2p}{p^2+1}\bar{\phi}(p) = \frac{1}{p^2}$$

$$\bar{\phi}(p) = \frac{p^2+1}{p^2(p-1)^2} = \frac{1}{(p-1)^2} + \frac{1}{p^2(p-1)^2} \quad \text{— (2)}$$

Taking inverse Laplace transform of both sides of (2), We have.

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$$\phi(x) = L^{-1}\left[\frac{1}{(p-1)^2}\right] + L^{-1}\left[\frac{1}{p^2(p-1)^2}\right]$$

$$= L^{-1}\left(\frac{1}{(p-1)^2}\right] + L^{-1}\left\{\frac{1}{p^2}\left\{1+2p+\frac{3p^2-2p^3}{(p-1)^2}\right\}\right]$$

$(p-1)^2 = 1-2p+p^2$

$$1-2p+p^2 \overline{)\,1\,}\, 1+2p$$
$$\underline{1-2p+p^2}$$
$$2p-p^2$$
$$\underline{2p-4p^2+2p^3}$$
$$3p^2-2p^3$$

$$= L^{-1}\left[\frac{1}{(p-1)^2}\right] + L^{-1}\left(\frac{1}{p^2}\right) + 2L^{-1}\left(\frac{1}{p}\right) + 3L^{-1}\left(\frac{1}{(p-1)^2}\right) - 2L^{-1}\left[\frac{p-1+1}{(p-1)^2}\right]$$

$$= \underline{L^{-1}\left(\frac{1}{(p-1)^2}\right]} + L^{-1}\left(\frac{1}{p^2}\right) + 2L^{-1}\left(\frac{1}{p}\right) + \underline{3L^{-1}\left(\frac{1}{(p-1)^2}\right]} - 2L^{-1}\left(\frac{1}{p-1}\right) - \underline{2L^{-1}\left(\frac{1}{(p-1)^2}\right]}$$

$$= 2L^{-1}\left[\frac{1}{(p-1)^2}\right] + x + 2 - 2e^x$$

$$= 2e^x . x + x + 2 - 2e^x$$

$\phi(x) = 2e^x(x-1) + x + 2$, which is the required soln.

(5) Solve $\phi(x) = e^{-x} - 2\int_0^x \cos(x-\xi)\,\phi(\xi)\,d\xi$

Solution. Given $\phi(x) = e^{-x} - 2\int_0^x \cos(x-\xi)\,\phi(\xi)\,d\xi$ — (1)

$$\phi(x) = e^{-x} - 2\cos x * \phi(x)$$

Taking Laplace transform of both sides of (1), we get

$$\bar{\phi}(p) = \frac{1}{p+1} - 2\left[L(\cos x) . L\{\phi(x)\}\right]$$

$$\bar{\phi}(p) = \frac{1}{p+1} - 2 . \frac{p}{p^2+1} . \bar{\phi}(p)$$

$$\left(1+\frac{2p}{p^2+1}\right)\bar{\phi}(p) = \frac{1}{p+1}$$

$$\frac{(p+1)^2}{p^2+1}\bar{\phi}(p) = \frac{1}{p+1}$$

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$$\bar{\phi}(p) = \frac{p^2+1}{(p+1)^3} = \frac{(p+1-1)^2+1}{(p+1)^3}.$$

Taking inverse Laplace transform

$$\phi(x) = e^{-x} L^{-1}\left[\frac{(p-1)^2+1}{p^3}\right] = e^{-x} L^{-1}\left[\frac{p^2-2p+2}{p^3}\right]$$

$$\phi(x) = e^{-x} L^{-1}\left[\frac{1}{p} - \frac{2}{p^2} + \frac{2}{p^3}\right]$$

$$\phi(x) = e^{-x}\left[1 - 2x + \cancel{2}\cdot\frac{x^2}{\cancel{\lfloor 2}}\right]$$

$$\phi(x) = e^{-x}(1-2x+x^2)$$

Q6 Solve $\phi(x) = c\sin x - 2\int_0^x \cos(x-\xi)\phi(\xi)d\xi$

Solution. Given $\phi(x) = c\sin x - 2\int_0^x \cos(x-\xi)\phi(\xi)d\xi$ —— (1)

$$\phi(x) = c\sin x - 2[\cos x * \phi(x)]$$

Taking Laplace transform of both sides of (1), we get

$$\bar{\phi}(p) = c\cdot\frac{1}{p^2+1} - \frac{2p}{p^2+1}\bar{\phi}(p)$$

$$\bar{\phi}(p)\left(1+\frac{2p}{p^2+1}\right) = \frac{c}{p^2+1} \Rightarrow \bar{\phi}(p) = \frac{c\,\cancel{(p^2+1)}}{(p+1)^2\cancel{(p^2+1)}}$$

Taking inverse Laplace transform, we get

$$\boxed{\phi(x) = c.\,e^{-x}.\,x}$$

Q7 Solve

$$\phi'(x) = x + \int_0^x \cos\xi\,\phi(x-\xi), \quad \phi(0) = 4$$

Solution. Given $\phi'(x) = x + \int_0^x \cos\xi\,\phi(x-\xi)$, $\phi(0) = 4$ —— (1)

$$\phi'(x) = x + \cos x * \phi(x) \quad \text{—— (2)}$$

Taking Laplace transform, we have

$$p\bar{\phi}(p) - \phi(0) = \frac{1}{p^2} + \frac{p}{p^2+1}\bar{\phi}(p) \quad [\because L[\phi'(x)] = (p\bar{\phi}(p) - \phi(0)]$$

$$\Rightarrow \left(p - \frac{p}{p^2+1}\right)\bar{\phi}(p) = \frac{1}{p^2} + 4$$

$$\Rightarrow \frac{p^2+\cancel{1}-\cancel{1}}{p^2+1}\bar{\phi}(p) = \frac{1}{p^3} + \frac{4}{p}$$

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$$\bar{\phi}(p) = \frac{p^2+1}{p^2} \times \frac{1}{p^3} + \frac{p^2+1}{p^2} \times \frac{4}{p}$$

$$\bar{\phi}(p) = \frac{1}{p^3} + \frac{1}{p^5} + \frac{4}{p} + \frac{4}{p^3}$$

$$\bar{\phi}(p) = \frac{4}{p} + \frac{5}{p^3} + \frac{1}{p^5}$$

Taking inverse Laplace transform, we have

$$\phi(x) = 4 + \frac{5x^2}{2} + \frac{x^4}{L4}$$

Q8, Solve $\phi'(x) = \sin x + \int_0^x \cos\xi \cdot \phi(x-\xi)\, d\xi$, $\phi(0)=0$

Soln.

Given $\phi'(x) = \sin x + \int_0^x \cos\xi \cdot \phi(x-\xi)\, d\xi$ ———

$\Rightarrow \phi'(x) = \sin x + \cos x * \phi(x)$ ——— (1)

$, \phi(0) = 0$ ——— (2)

Taking Laplace transform of both sides of (1), we get

$$L[\phi'(x)] = L(\sin x) + L(\cos x * \phi(x))$$

$$p\bar{\phi}(p) - 0 = \frac{1}{p^2+1} + \frac{p}{p^2+1} \cdot \bar{\phi}(p) \quad [\because L(\cos x * \phi(x)) = L(\cos x) \cdot L[\phi(x)]]$$

$$\left(p - \frac{p}{p^2+1}\right)\bar{\phi}(p) = \frac{1}{p^2+1}$$

$$p\left(\frac{p^2+1-1}{p^2+1}\right)\bar{\phi}(p) = \frac{1}{p^2+1}$$

$$\Rightarrow \bar{\phi}(p) = \frac{1}{p^3}$$

Taking inverse Laplace transform, we have

$$\phi(x) = L^{-1}\left[\frac{1}{p^3}\right]$$

$$\boxed{\phi(x) = \frac{x^2}{2}}$$, which is required Solution.